

Linearity rules of integration

Introduction

To enable us to find integrals of a wider range of functions than those normally given in a Table of Integrals we can make use of two rules known as **linearity rules**.

1. The integral of a constant multiple of a function

A constant factor in an integral can be moved outside the integral sign in the following way.

$$\int k f(x) dx = k \int f(x) dx$$

This is only possible when k is a constant, and it multiplies some function of x .

Example

Find $\int 11x^2 dx$.

Solution

We are integrating a multiple of x^2 . The constant factor, 11, can be moved outside the integral sign.

$$\int 11x^2 dx = 11 \int x^2 dx = 11 \left(\frac{x^3}{3} + c \right) = \frac{11x^3}{3} + 11c$$

where c is the constant of integration. Because $11c$ is a constant we would normally write the answer in the form $\frac{11x^3}{3} + K$ where K is another constant.

Example

Find $\int -5 \cos x dx$.

Solution

We are integrating a multiple of $\cos x$. The constant factor, -5 , can be moved outside the integral sign.

$$\int -5 \cos x dx = -5 \int \cos x dx = -5 (\sin x + c) = -5 \sin x + K$$

where K is a constant.

2. The integral of the sum or difference of two functions

When we wish to integrate the sum or difference of two functions, we integrate each term separately as follows:

$$\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$$
$$\int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$$

Example

Find $\int (x^3 + \sin x) \, dx$.

Solution

$$\int (x^3 + \sin x) \, dx = \int x^3 \, dx + \int \sin x \, dx = \frac{x^4}{4} - \cos x + c$$

Note that only a single constant of integration is needed.

Example

Find $\int e^{3x} - x^7 \, dx$.

Solution

$$\int e^{3x} - x^7 \, dx = \int e^{3x} \, dx - \int x^7 \, dx = \frac{e^{3x}}{3} - \frac{x^8}{8} + c$$

Exercises

1. a) Find $\int 8x^5 + 3x^2 \, dx$, b) $\int \frac{2}{3}x \, dx$.
2. Find $\int 3 \cos x + 7x^3 \, dx$.
3. Find $\int 7x^{-2} \, dx$.
4. Find $\int \frac{5}{x} \, dx$.
5. Find $\int \frac{x + \cos 2x}{3} \, dx$.
6. Find $\int 5e^{4x} \, dx$.
7. Find $\int \frac{e^x - e^{-x}}{2} \, dx$.

Answers

1. a) $\frac{4x^6}{3} + x^3 + c$, b) $\frac{1}{3}x^2 + c$, 2. $3 \sin x + \frac{7x^4}{4} + c$, 3. $-\frac{7}{x} + c$ 4. $5 \log_e |x| + c$,
5. $\frac{x^2}{6} + \frac{\sin 2x}{6} + c$, 6. $\frac{5e^{4x}}{4} + c$, 7. $\frac{e^x + e^{-x}}{2} + c$.